

UNIRULEDNESS OF STRATA OF HOLOMORPHIC DIFFERENTIALS IN SMALL GENUS

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ABSTRACT. We address the question about the birational geometry of the strata of holomorphic differentials. Using pencils on K3 surfaces in small genera we can construct rational curves on strata of holomorphic differentials and prove uniruledness in several cases. Unirationality is proven for some partition range in small genera. Using pencils on del Pezzo surfaces we prove uniruledness of strata of quadratic differentials for small genus.

INTRODUCTION

Let $g \geq 2$ be an integer and $\mu = (m_1, \dots, m_n)$ be a partition of $2g - 2$. The moduli space of canonical divisors of type μ is defined as the closed substack $\mathcal{H}_g(\mu) \subset \mathcal{M}_{g,n}$ given by

$$\mathcal{H}_g(\mu) = \left\{ [C, p_1, \dots, p_n] \in \mathcal{M}_{g,n} \mid \mathcal{O}_C \left(\sum_{i=1}^n m_i p_i \right) \cong \omega_C \right\}.$$

To some extent Diaz in [Dia84], and more precisely and in a greater generality Polischuck in [Pol06], proved that $\mathcal{H}_g(\mu)$ is nonsingular and of pure codimension $g - 1$ when all terms of μ are positive. The same arguments apply if we allow the partition μ to have negative terms, in which case $\mathcal{H}_g(\mu)$ is again nonsingular and of pure codimension g . Kontsevich and Zorich [KZ03] showed that when the entries of the partition are all positive, $\mathcal{H}_g(\mu)$ may have up to three connected components depending on the parity of the entries and the length of the partition. For instance when all entries of μ are even,

$$\eta = \frac{1}{2} \sum m_i x_i$$

is a theta characteristic and the parity (i.e. the parity of $h^0(C, \eta)$) is stable under deformations. See [Mum71]. When $\mu = (2, \dots, 2)$, there is a dominant forgetful map

$$\begin{aligned} \mathcal{H}_g(\mu) &\rightarrow \mathcal{S}_g^- \amalg Z \\ [C, x_1, \dots, x_{g-1}] &\mapsto [C, x_1 + \dots + x_{g-1}] \end{aligned}$$

where \mathcal{S}_g^- is the space of odd theta characteristics and $Z \subset \mathcal{S}_g^+$ is the divisor of even theta characteristics with at least two sections. In this case our space breaks into at least two components.

We would like to study the global geometry of the Zariski closure

$$\overline{\mathcal{H}}_g(\mu) \subset \overline{\mathcal{M}}_{g,n}.$$

In some sense a modular interpretation of the boundary $\partial \overline{\mathcal{H}}_g(\mu)$ was recently given. G. Farkas and R. Pandharipande [FP15] defined a proper moduli space called moduli space of *twisted canonical divisors* and it contains our space of interest as an open subset. This was the first big step in the understanding of how to compactify $\mathcal{H}_g(\mu)$. The compactification problem was carried further by M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky and M. Möller

TABLE 1.

	<i>Unirational</i>	<i>Uniruled</i>
$3 \leq g \leq 6$	$l(\mu) \leq g - 1$	<i>No restriction on μ</i>
$g = 7, 8$?	<i>No restriction on μ</i>
$g = 9$?	$l(\mu) \geq 7$
$g = 10$?	$11 \leq l(\mu) < 18$
$g = 11$?	$l(\mu) \geq 10$

in [BCG⁺]. Unlike Farkas and Pandharipande's space we cannot decide whether a stable curve lies on the boundary of the strata by studying its dual graph together with a set of equations in the jacobian of each component. The boundary components are parametrized not just by the topological type of the nodal curves and strata in smaller genera, but also by their particular complex structure incarnated in residue conditions for the nodes.

The next question that comes naturally in the study of $\overline{\mathcal{H}}_g(\mu)$ is about its birational geometry. The main aim of this paper is to establish a range for the genus and length of partition where the Kodaira dimension is negative. When the partition is positive (we call it *holomorphic*) the space that serves us as model and highlights the change from uniruledness to general type should be the moduli of odd spin curves $\overline{\mathcal{S}}_g^-$. We follow the method of [FV14] used to prove uniruledness and unirationality of the space of odd theta characteristics in small genera.

The idea of the proof goes as follows, when the length of μ is greater than $g - 1$ the forgetful map $\mathcal{H}_g(\mu) \rightarrow \mathcal{M}_g$ is dominant and for small enough genus the general smooth curve can be realized as a canonical curve given by a hyperplane section of a K3 surface $S \subset \mathbb{P}^g$. Since the divisor $\sum m_i x_i$ is canonical, is the intersection of a codimension 2 plane with S . The rational map is given by the pencil spanned by this plane in \mathbb{P}^g .

The genus $g = 10$ is especially delicate and we treat it by studying 1-nodal models of geometric genus 10 curves lying on a K3 surface. For generic choice we proved that we can always find such models by deformation theory arguments that reduce the question of dominance of a moduli map to a cohomology computation.

To avoid tedious repetition, in what follows we restrict our selves to the case where μ is an holomorphic partition of $2g - 2$ and $g \geq 3$.

Theorem 0.1. *Let $\overline{\mathcal{H}}_g(\mu)$ be an irreducible stratum with length of partition $l(\mu)$. We have uniruledness and unirationality as in Table 0.1.*

In [KZ03] are listed the possible partitions where the strata are not irreducible.

- When $\mu = (2g - 2)$ or $\mu = (2l, 2l)$ then $\mathcal{H}_g(\mu)$ has three connected components; the hyperelliptic one \mathcal{H}^{hyp} and the even and odd \mathcal{H}^+ , \mathcal{H}^- depending on the parity of the induced theta characteristic, i.e., $[C, x_1, \dots, x_n] \in \mathcal{H}^+$ (resp. \mathcal{H}^-) if and only if $h^0\left(C, \mathcal{O}_C\left(\frac{1}{2}\sum m_i x_i\right)\right) \equiv 0$ (resp. $\equiv 1$) mod 2.
- When $\mu = (2l_1, \dots, 2l_r)$ with $r \geq 3$ then $\mathcal{H}_g(\mu)$ has two connected components depending on the parity as before \mathcal{H}^+ and \mathcal{H}^- .
- Finally when $\mu = (2l - 1, 2l - 1)$ then $\mathcal{H}_g(\mu)$ has two connected components the hyperelliptic one \mathcal{H}^{hyp} and a complementary one \mathcal{H}^{nonhyp} .

Theorem 0.2. *For every genus $\mathcal{H}^{hyp}(2g-2)$ is unirational and even and odd strata are uniruled for any partition in the range $g \leq 8$. In genus $g = 9, 11$ the odd stratum \mathcal{H}^- is uniruled for partition $\mu = (2, \dots, 2)$ and the S_{g-1} -quotient of the even stratum \mathcal{H}^+ for the same partition is uniruled in every genus.*

Of equal interest is the strata of k -differentials when $k \geq 2$. In section 2 we treat the case $k = 2$. Let $\nu = (n_1, \dots, n_{l(\nu)})$ be a partition of $4g - 4$ with length $l(\nu)$ and

$$\mathcal{Q}(\nu) = \left\{ \left[C, p_1, \dots, p_{l(\nu)} \right] \in \mathcal{M}_{g, l(\nu)} \mid \mathcal{O}_C \left(\sum_{i=1}^{l(\nu)} n_i p_i \right) \cong \omega_C^{\otimes 2} \right\}$$

the strata of quadratic differentials. The space $\mathcal{Q}(\nu)$ might have at most three connected components coming from parity of theta characteristics when 2 divides every entry of ν and hyperelliptic curves. See [Lan02]. The dimension of $\mathcal{Q}(\nu)$ when at least one entry is non-even is $2g - 3 + l(\nu)$. See [Vee90] or [Sch, Thm. 1.1]. Instead of K3 surfaces we use del Pezzo surfaces to construct rational curves on $\mathcal{Q}(\nu)$.

Theorem 0.3. *For genus $3 \leq g \leq 6$ and $l(\nu) \geq g$, the moduli space of quadratic differentials $\mathcal{Q}(\nu)$ is uniruled.*

Nodal Curves on K3 Surfaces. A K3 surface S is a smooth complex projective variety with $K_S \cong \mathcal{O}_S$ and $h^1(S, \mathcal{O}_S) = 0$. A polarized K3 surface of genus g is a pair (S, H) where H is a nef line bundle on S , $|H|$ has no fixed components and $H^2 = 2g - 2$ (hence big for $g \geq 2$). Such pairs form a smooth moduli space denoted \mathcal{F}_g . Given such a pair $(S, H) \in \mathcal{F}_g$, the linear system $|H|$ is base point free (see [Tan82, thm. 2.1]) of dimension g and if it does not contain hyperelliptic curves, the induced map

$$\psi_{|H|} : S \rightarrow \mathbb{P}^g$$

is birational onto the image. Its image is a degree $2g - 2$ surface on \mathbb{P}^g , with at worst canonical singularities and general hyperplane sections are canonical curves of genus g . Moreover, we have also the converse. See [Rei97, Ch. 3.3] or [Sai74].

Given a positive integer $\delta \leq g$, the Severi variety of irreducible δ -nodal curves in the linear system $|H|$ is denoted by $V_\delta(S, H)$. It is well known since Mumford (see e.g. [Tan82] or [Fla01]) that for $(S, H) \in \mathcal{F}_g$ general and $g \geq 2$, $V_\delta(S, H)$ is non-empty and regular, moreover each irreducible component is of dimension $g - \delta$.

Definition 0.4. For g, δ with $g \geq 3$ and $0 \leq \delta \leq g - 2$, we define the universal Severi variety $\mathcal{V}_{g, \delta}$ to be the algebraic stack of pairs (S, X) with $(S, H) \in \mathcal{F}_g$ and $X \in V_\delta(S, H)$.

The stack $\mathcal{V}_{g, \delta}$ is smooth and every irreducible component has dimension $19 + (g - \delta)$. It was conjectured by C. Ciliberto and T. Dedieu [CD12, Thm. 2.1] that $\mathcal{V}_{g, \delta}$ is irreducible and proved in the range $3 \leq g \leq 11$, $g \neq 10$ and $0 \leq \delta \leq g$. They conjectured that $\mathcal{V}_{g, \delta}$ is always irreducible. The natural forgetful map

$$\pi_\delta : \mathcal{V}_{g, \delta} \rightarrow \mathcal{F}_g$$

is smooth and restricted to any irreducible component is dominant. See [FKPS08].

There is a well defined moduli map

$$c_{g, \delta} : \mathcal{V}_{g, \delta} \rightarrow \mathcal{M}_{g - \delta}$$

sending a pair (S, X) to the isomorphism class of the normalization of X . When $\delta = 0$ this is Mukai's map, dominant for $g \leq 9$ and $g = 11$, not dominant for $g = 10$ and generically finite over the image for $g \geq 11$ and $g \neq 12$. See [Muk88], [Muk96] and [Muk92].

A. Beauville in [Bea04, §5] studied the differential of $c_{g,0}$ and gave a deformation theoretic proof of Mukai's results. A. Beauville, F. Flamini, A. L. Knutsen, G. Pacienza and E. Sernesi proved in [FKPS08] that in the range $3 \leq g \leq 11$ and $0 \leq \delta \leq g-2$, if $V \subset \mathcal{V}_{g,\delta}$ any irreducible component, then for $2 \leq g-\delta < g \leq 11$ the restricted map

$$c_{g,\delta}|_V: V \rightarrow \mathcal{M}_{g-\delta}$$

is dominant with general fiber of dimension $22-2(g-\delta)$. Let $(S, X) \in \mathcal{V}_{g,\delta}$ be a δ -nodal curve on a K3 and $\nu: C \rightarrow X$ the normalization map with $\nu^{-1}(\text{Sing}(X)) = \{p_1, q_1, \dots, p_\delta, q_\delta\}$. We can consider the following moduli map

$$\begin{aligned} c: \mathcal{V}_{g,\delta} &\rightarrow \mathcal{M}_{g-\delta, [2\delta]} \\ (S, X) &\mapsto [C, p_1 + q_1 + \dots + p_\delta + q_\delta]. \end{aligned}$$

The following theorem is the main ingredient to treat the genus 10 case. Following [FKPS08] we were able to extend their result.

Theorem 0.5. *In the range $3 \leq g \leq 11$, $1 \leq \delta \leq g-2$ and $g \neq 10$, the moduli map*

$$c: \mathcal{V}_{g,\delta} \rightarrow \mathcal{M}_{g-\delta, [2\delta]}$$

is dominant with general fiber dimension $22-2g$. Moreover when $g = 10$, the image of c has always codimension one and its class in $\text{Pic}_{\mathbb{Q}}(\mathcal{M}_{10-\delta, [2\delta]})$ is given by

$$c_*[\mathcal{V}_{10,\delta}] = \frac{1}{2\delta!}(7\lambda + \psi).$$

Here the class ψ is defined to be the push forward on the S_n -invariant cycle $\psi_1 + \dots + \psi_{2\delta}$ by the finite quotient map $\mathcal{M}_{g-\delta, 2\delta} \rightarrow \mathcal{M}_{g-\delta, [2\delta]}$.

G. Farkas and M. Popa [FP05, Thm 1.6] proved that the class of the closure $\overline{\mathcal{K}} \in \text{Pic}(\overline{\mathcal{M}}_{10})$ of the locus of smooth curves lying on a K3 surface is given by

$$\overline{\mathcal{K}} = 7\lambda - \delta_0 - 5\delta_1 - 9\delta_2 - 12\delta_3 - 14\delta_4 - B_5\delta_5$$

with $B_5 \geq 6$. The formula for $c_*[\mathcal{V}_{10,\delta}]$ is a pull back of $\overline{\mathcal{K}}$ by a boundary morphism. See [ACG11, Chapter 17, Lemma 4.35].

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1. UNIRULEDNESS OF STRATA.

As explained in the introduction, the general strategy is to construct pencils on K3 surfaces to prove uniruledness.

Lemma 1.1. *Let $(S, H) \in \mathcal{F}_g$ be a polarized K3 of genus g and $P \subset |H|$ a pencil whose general element is a smooth genus g curve. Then the induced family of curves on S is not isotrivial.*

Proof. Let $Z = \text{Bs}(P)$ be the base locus of the pencil and \mathcal{U} the family of genus g curves on S . Recall Castelnuovo's theorem saying that on a smooth surface a curve E can be contracted by a birational map to a regular point if, and only if, E is rational with self intersection $E^2 = -1$. Then \mathcal{U}, S and P sit in the following diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\text{Bl}_Z S} & S \\ \pi \downarrow & \swarrow & \\ P & & \end{array}$$

The fundamental group of $P \cong \mathbb{P}^1$ is trivial and therefore acts trivially on the homology of the fiber. Thus, if π has no singular fiber then

$$\chi(\mathcal{U}, \mathbb{Z}) = \chi(P, \mathbb{Z}) \cdot \chi(\pi^{-1}(\text{point}), \mathbb{Z}) = 2 \cdot (2 - 2g)$$

On the other hand, the topological Euler characteristic of a K3 is 24 and we need to add the number of points that we blew up

$$24 + |Z| = 2(2 - 2g),$$

which is a contradiction. Then $\mathcal{U} \rightarrow P$ has singular fibers, so in particular cannot be isotrivial. \square

When $(S, H) \in \mathcal{F}_g$ is general, $\text{Pic}(S) = \mathbb{Z}H$ and therefore, every singular curve on $|H|$ is irreducible. Thus the induced moduli map $m(\pi) : P \rightarrow \overline{\mathcal{M}}_g$ is never trivial. Moreover if P is a general pencil in $|H|$, it has $2g - 2$ base points. We blow-up to get a fibration $\mathcal{U} \rightarrow P$ with general fiber F a smooth genus g curve. The singular fibers are irreducible nodal curve and from the relation

$$\chi(\mathcal{U}, \mathbb{Z}) = \chi(P, \mathbb{Z}) \cdot \chi(F, \mathbb{Z}) + \text{number of singular fibers}$$

one deduce that $P \cdot \Delta_{\text{irr}} = 6g + 18$, i.e., the hypersurface parametrizing singular curves $D_{S,H} \subset |H|$ has degree $6g + 18$.

1.1. General Case, $g \leq 11$ and $g \neq 10$. Recall that a variety X is uniruled if for a general point $p \in X$ there is a rational curve passing through it. In other words if there exist a variety Y and a dominant rational map $Y \times \mathbb{P}^1 \dashrightarrow X$. Let $\mu = (m_1, \dots, m_n)$ be an holomorphic partition of $2g - 2$ with length n and

$$[C, x_1, \dots, x_n] \in \mathcal{H}_g(\mu)$$

a point on the stratum. Assume $3 \leq g \leq 9$ or $g = 11$. The forgetful map $\pi : \mathcal{H}_g(\mu) \rightarrow \mathcal{M}_g$ is dominant when the length of the partition is greater or equal than $g - 1$. See [Gen]. The curve C is general and therefore can be embedded as a hyperplane section on a genus g polarized K3 surface $[S, H] \in \mathcal{F}_g$. See [Muk88] and [Muk96]. Here \mathcal{F}_g is the moduli space of principally polarized K3 surfaces $[S, H]$, where S is a K3 surface and $H \in \text{Pic}(S)$ is a (primitive) polarization of degree $H^2 = 2g - 2$. We construct a rational curve

$$\mathbb{P}^1 \rightarrow \overline{\mathcal{H}}_g(\mu)$$

passing through $[C, x_1, \dots, x_n]$. Our curve is embedded in S as hyperplane section

$$C \cong S \cap H \hookrightarrow S \subset \mathbb{P}H^0(S, H)^\vee \cong \mathbb{P}^g.$$

The divisor $m_1x_1 + \dots + m_nx_n \in \text{Div}(C)$ is canonical so can be realized as a hyperplane section of $H \cong \mathbb{P}^{g-1}$, i.e., a point $\Lambda_\mu \in \mathbb{G}(g-2, \mathbb{P}^g)$ such that

$$\Lambda_\mu \cdot S = m_1x_1 + \dots + m_nx_n.$$

Let

$$P \cong \{H' \in (\mathbb{P}^g)^\vee \mid \Lambda_\mu \subset H'\}$$

be the pencil of hyperplanes containing Λ_μ in \mathbb{P}^g . Since $C \in P$ is smooth, for a general hyperplane $H' \in P$, the curve $C' = H' \cap S \hookrightarrow H' \cong \mathbb{P}^{g-1}$ is smooth and canonically embedded. Moreover, the hyperplane $\Lambda_\mu \subset \mathbb{P}^{g-1}$ is a canonical divisor of the form

$$\Lambda_\mu \cdot S = \Lambda_\mu \cdot C = m_1x_1 + \dots + m_nx_n.$$

This construction gives us a map defined on an open subset of $P \cong \mathbb{P}^1$

$$\begin{aligned} \gamma : \mathbb{P}^1 &\dashrightarrow \overline{\mathcal{H}}_g(\mu) \\ H' &\mapsto [H' \cap S, x_1, \dots, x_n]. \end{aligned}$$

The map can be extended and we already prove in Lemma 1.1 that cannot be trivial. This give us Theorem 0.1 for $g \leq 9$ and $g = 11$ when the length of μ is at least $g-1$.

1.2. Special Cases for Genus $g \leq 8$. Let C be a smooth curve of genus $g \geq 2$. An *ample K3 extension* of C is a K3 surface S with at worst rational double points which contains C in the smooth locus as ample divisor. Let us recall a result of Ide [Ide08].

Theorem 1.2. *All smooth curves of genera $2 \leq g \leq 8$ have ample K3 extensions. Moreover, they have smooth ample K3 extensions except in the following cases;*

- $g = 6, 7, 8$ and $K_C = 2D$ where D is a g_{g-1}^2 , or
- $g = 8$ and $K_C = A + 2B$ where A is a g_4^1 and B is a g_5^1 .

Every complex surface S with at worst ADE singularities admits a crepant resolution (see [Rei85]). In our case S is a K3 surface with at worst rational double points, so there is a unique resolution

$$\pi : \tilde{S} \rightarrow S.$$

The resolution is crepant meaning $\pi^*K_S = K_{\tilde{S}} = 0$ and $h^1(\mathcal{O}_{\tilde{S}})$ is a birational invariant for surfaces with mild enough singularities. The smooth surface \tilde{S} is again a K3 and if \tilde{C} is the proper transform of C , as divisor \tilde{C} might cease to be ample (it can have trivial intersection with -2 -curves) but the self intersection is positive; it is still big and nef. We can rephrase Ide's theorem as follow.

Theorem 1.3. *Every smooth curve C of genus $2 \leq g \leq 8$ can be embedded in a smooth K3 surface S with $C \subset S$ big and nef.*

Let $[C, x_1, \dots, x_n] \in \mathcal{H}_g(\mu)$ be a general point on the stratum with $3 \leq g \leq 8$, $\mathcal{H}_g(\mu)$ connected and S be a big and nef K3 extension of C . Then the map $\phi_C : S \dashrightarrow \hat{S} \subset \mathbb{P}^g$ restricted to C is the canonical map

$$\phi_C|_C : C \rightarrow \mathbb{P}^{g-1}.$$

The point $[C, x_1, \dots, x_n]$ is general and we are under the assumption that $\mathcal{H}_g(\mu)$ is connected. Therefore, C is not hyperelliptic and ϕ_{K_C} is an embedding. We can repeat the same construction as for the general case. Since the general hyperplane of \mathbb{P}^g in the pencil of hyperplanes

through Λ_μ as before intersects \hat{S} in a smooth curve, the pull back is smooth (it does not contain -2 -curves).

This give us uniruledness for every irreducible stratum $\overline{\mathcal{H}}_g(\mu)$ in the range $3 \leq g \leq 8$.

1.3. Genus $g = 9$. For small length partitions the forgetful map $\pi : \mathcal{H}_g(\mu) \rightarrow \mathcal{M}_g$ is no longer dominant and to carry out the argument above one has to prove that the image of Mukai's map $\mathcal{F}_g \rightarrow \mathcal{M}_g$ intersected with $\pi(\mathcal{H}_g(\mu))$ contains a non empty open in $\pi(\mathcal{H}_g(\mu))$. A smooth complex curve of genus 9 can be realized as an hyperplane section of a K3 if is not pentagonal (has no g_5^1). See [Muk10]. In particular the image of $\mathcal{F}_9 \rightarrow \mathcal{M}_9$ contains the complement of the Brill-Noether divisor D_5^1 of pentagonal curves. S. Mullane [Mul15, §5] computed the class of the closure of the image $\mathcal{H}_g(\mu) \rightarrow \mathcal{M}_g$ when $l(\mu) = g - 2$. In genus 9 the slope is strictly bigger than the Brill-Noether Slope. In any case, when connected, $\pi(\mathcal{H}_9(\mu)) \subset \mathcal{M}_9$ is not contained in the pentagonal locus, therefore the general point can be embedded in a K3 surfaces and the argument above can still be carried out.

Remark 1.4. By specialization, it would be enough to show that there is a Brill-Noether general curve in the non-hyperelliptic component of $\mathcal{H}_g(2g - 2)$, but to construct a Brill-Noether general curve C admitting a subcanonical point is not an easy task.

1.4. Genus 10. The genus 10 is much more delicate and it is done by studying irreducible nodal curves of genus 11. We define the open set $\mathcal{U}_{10} \subset \mathcal{H}_{10}(\mu)$ by the condition

$$[C, x_1, \dots, x_n] \in \mathcal{U}_{10}$$

if and only if there exist a polarized K3 surface $[S, H] \in \mathcal{F}_{11}$ and a non trivial map $f : C \rightarrow S$, such that $f_*[C] \in |H|$ and f is the normalization map of the irreducible nodal curve $f(C)$ having a single node at $f(x_1) = f(x_2)$.

Proposition 1.5. *Every component of $\overline{\mathcal{U}}_{10} \subset \overline{\mathcal{H}}_{10}(\mu)$ is uniruled.*

Proof. Let $[C, x_1, \dots, x_n]$ be a general point on \mathcal{U}_{10} and

$$\epsilon : \tilde{S} \rightarrow S$$

the blow up of S at the node $f(x_1) = f(x_2)$. Then C can be embedded in \tilde{S} , moreover $C \in |\epsilon^*H - 2E|$, where E is the exceptional divisor of ϵ . By adjunction

$$\mathcal{O}_C(C) \cong K_C(-x_1 - x_2) \cong \mathcal{O}_C \left((m_1 - 1)x_2 + (m_2 - 1)x_2 + \sum_{i \geq 3} m_i x_i \right).$$

let us assume that $l(\mu) \geq 2$ the divisor

$$D = (m_1 - 1)x_2 + (m_2 - 1)x_2 + \sum_{i \geq 3} m_i x_i$$

is effective on C . The following sequence is exact

$$0 \rightarrow \mathcal{O}_{\tilde{S}} \rightarrow \mathcal{I}_{D/\tilde{S}}(C) \rightarrow \mathcal{O}_C \rightarrow 0,$$

where the middle term is the ideal sheaf of the closed subscheme $D \subset \tilde{S}$ twisted by C . Since \tilde{S} is simply connected (recall that $h^{0,1}$ is a birational invariant of smooth surfaces)

$$\mathbb{P}H^0 \left(\tilde{S}, \mathcal{I}_{D/\tilde{S}}(C) \right) \cong \mathbb{P}^1.$$

There is a rational map

$$\mathbb{P}^1 \dashrightarrow \mathcal{U},$$

sending the generic element $C' \in |\varepsilon^* H - 2E|$ passing through $\text{Supp}(D)$ to

$$C' \in \mathbb{P}^1 \mapsto [C', x_1, x_2, \dots, x_n].$$

The same argument as in Lemma 1.1 applies to prove non-isotriviality. We might assume $m_1 \geq m_2 \geq \dots \geq m_n$. Notice that the argument fails when the set $\{x_1, x_2\}$ is not contained in the support of D . If $m_1 > m_2 = 1$ we can still keep track of the points since $x_1 \in \text{Supp}(D)$ and for C'' general, $C'' \cap E = x_1 + q$. We impose $x_2 = q$ and the argument still holds true. \square

When $m_2 = m_1 = 1$, then $\mu = (1, \dots, 1)$ with $l(\mu) = 18$ and our map is well defined in the quotient

$$\begin{aligned} \mathbb{P}^1 &\dashrightarrow \mathcal{U}_{10}/(\mathbb{Z}/2\mathbb{Z}) \\ C'' &\mapsto [C'', y_1 + y_2, x_3, \dots, x_n] \end{aligned}$$

where $y_1 + y_2 = C'' \cap E$ and we have uniruledness for the quotient

$$\mathcal{U}_{10} \rightarrow \mathcal{U}_{10}/(\mathbb{Z}/2\mathbb{Z}).$$

Consider the cartesian diagram

$$\begin{array}{ccc} \mathcal{V}_{11,1} \times \mathcal{H}_{10}(\mu) & \xrightarrow{p_2} & \mathcal{H}_{10}(\mu) \\ p_1 \downarrow & & \downarrow \pi \\ \mathcal{V}_{11,1} & \xrightarrow{c} & \mathcal{M}_{10,[2]}. \end{array}$$

Notice that the image of p_2 is exactly \mathcal{U}_{10} . And in the range $l(\mu) \geq g + 1$ the map π is dominant. It remains to prove that c is dominant to conclude that \mathcal{U}_{10} dominates the strata and therefore uniruledness follows.

Corollary 1.6 (of Theorem 0.5). *For every holomorphic partition μ of length $18 > l(\mu) \geq 11$ the inclusion defined above $\mathcal{U}_{10} \hookrightarrow \mathcal{H}_{10}(\mu)$ is dominant.*

Corollary 1.7 (of Theorem 0.5). *For every partition of length $l(\mu) \geq 11$, the set $\mathcal{U}_{10} \subset \overline{\mathcal{H}}_{10}(\mu)$ is non-empty and contains an open dense subset. In particular $\overline{\mathcal{H}}_{10}(\mu)$ is uniruled for $11 \leq l(\mu) < 18$.*

In the coming section we give a proof of Theorem 0.5.

1.5. Deformation theory of nodal curves on K3 surfaces. We recall a few facts about deformation theory of nodal curves on K3 surfaces. For general reference we refer to [Ser06] and specific to our situation to [FKPS08].

Locally trivial deformations of the pair (S, X) are governed by the sheaf $T_S\langle X \rangle$ defined to be the preimage of $T_X \subset T_S|_X$ under restriction $T_S \rightarrow T_S|_X$. It sits in two standard exact sequences

$$0 \rightarrow T_S(-X) \rightarrow T_S\langle X \rangle \rightarrow T_X \rightarrow 0$$

and

$$0 \rightarrow T_S\langle X \rangle \rightarrow T_S \rightarrow N'_{X/S} \rightarrow 0$$

where $N'_{X/S}$ is the *equisingular normal sheaf* of X in S . This sheaf governs the deformation theory when S is fixed. Moreover,

$$T_X V_\delta(S, H) = H^0(X, N'_{X/S}).$$

The first order locally trivial deformations of the pair (S, X) are parametrized by $H^1(S, T_S\langle X \rangle)$. Obstructions are parametrized by H^2 and local automorphisms by H^0 . The theory is unobstructed and the coarse moduli $\mathcal{V}_{g,\delta}$ is smooth [FKPS08, Prop 4.8]. For any (S, X)

$$h^2(T_S\langle X \rangle) = h^0(T_S\langle X \rangle) = 0.$$

and

$$T_{(S,X)}\mathcal{V}_{g,\delta} = H^1(S, T_S\langle X \rangle).$$

Much more can be said. Given such a pair (S, X) there is a unique embedded resolution of X given by the following diagram

$$\begin{array}{ccccc} C \cap E & \hookrightarrow & C & \hookrightarrow & \tilde{S} \\ \downarrow & & \downarrow f & & \downarrow \varepsilon \\ \text{Sing}(X) & \hookrightarrow & X & \hookrightarrow & S, \end{array}$$

where \tilde{S} is the Blow-up of S along the nodes, $E = E_1 + \dots + E_\delta$ is the exceptional divisor, C is a smooth genus $g - \delta$ curve and $f : C \rightarrow X \subset S$ is the normalization map. Let us take a look at the tangent exact sequence for the normalization map

$$0 \rightarrow T_C \rightarrow f^*T_S \rightarrow N_f \rightarrow 0.$$

Here N_f is the normal sheaf of the map $f : C \rightarrow S$. With this notation [FKPS08, Lemma 4.16]

$$f_*(N_f) = N'_{X/S} \text{ and } H^i(N'_{X/S}) \cong H^i(N_f) \text{ for } i = 0, 1.$$

This is not surprising since the group $H^0(N_f)$ can be identified with the tangent space at $[f]$ of the space of maps $f : C \rightarrow S$ from genus $g - \delta$ smooth curves to a fixed target with $f_*C = H$. There is a one to one correspondence between δ -nodal curves on S in the linear system $|H|$ and maps f . The correspondence is given by normalization

$$\begin{array}{ccc} V_\delta(S, H) & \rightarrow & M_{g-\delta}(S, H) \\ X \subset S & \mapsto & f : C \rightarrow S. \end{array}$$

To recover the differential of the map $c_{g,\delta}$ we go to \tilde{S} . Consider the following diagram as in [FKPS08]

$$(1) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \varepsilon^*T_S(-C) & \xrightarrow{\cong} & \varepsilon^*T_S(-C) & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}_C & \longrightarrow & \varepsilon^*T_S & \xrightarrow{\lambda} & N_f \longrightarrow 0 \\ & & \downarrow \tau & & \downarrow & & \parallel \\ 0 & \longrightarrow & T_C & \longrightarrow & f^*T_S & \longrightarrow & N_f \longrightarrow 0, \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

where \mathcal{F}_C is the kernel of the composition $\lambda : \varepsilon^* T_S \rightarrow f^* T_S \rightarrow N_f$. Turns out [FKPS08, Prop. 4.22] that

$$H^i(S, T_S(X)) \cong H^i(\tilde{S}, \mathcal{F}_C) \text{ for } i = 0, 1, 2$$

and $H^1(\tau)$ is the differential of the map $c_{g,\delta}$. The four authors proved that in the desired range, for a general choice $H^1(\tau)$ is surjective, see [FKPS08, Thm. 5.1].

Proposition 1.8. *With notation as above*

$$H^i(\tilde{S}, \mathcal{F}_C(-E)) \cong H^i(\tilde{S}, \mathcal{F}_C) \text{ for } i = 0, 1, 2.$$

In particular $H^1(\mathcal{F}_C(-E))$ parametrizes locally trivial first order deformations of the pair (S, X) .

Proof. Consider

$$0 \rightarrow \mathcal{F}_C(-E) \rightarrow \mathcal{F}_C \rightarrow \mathcal{F}_C|_E \rightarrow 0.$$

Is enough to prove that

$$\mathcal{F}_C|_{E_i} \cong \mathcal{O}_{E_i}(-1)^{\oplus 2}.$$

Let $j : E_i \hookrightarrow \tilde{S}$ be the close embedding of one of the components of the exceptional divisor. Notice that $\varepsilon^* T_S|_{E_i} \cong \mathcal{O}_{E_i}^{\oplus 2}$ and $N_f \cong \omega_C$. From the second arrow in (1) we have

$$0 \rightarrow L^1 j^* \omega_C \rightarrow \mathcal{F}_C|_{E_i} \rightarrow \mathcal{O}_{E_i}^{\oplus 2} \rightarrow \mathcal{O}_{E_i \cap C} \rightarrow 0$$

and $C \cap E = p_i + q_i$. Then

$$\deg(\mathcal{F}_C|_{E_i}) = \deg L^1 j^* \omega_C - 2.$$

On the other hand by adjunction $\omega_C \cong \mathcal{O}_C(E + C)$ and pulling back by j the exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{S}}(E) \rightarrow \mathcal{O}_{\tilde{S}}(E + C) \rightarrow \omega_C \rightarrow 0$$

we get

$$0 \rightarrow L^1 j^* \omega_C \rightarrow \mathcal{O}_{E_i}(E_i) \rightarrow \mathcal{O}_{E_i}(E_i + C) \rightarrow \mathcal{O}_{p_i + q_i} \rightarrow 0.$$

Counting degrees $\deg \mathcal{O}_{E_i}(E_i) = -1$ and $\deg \mathcal{O}_{E_i}(E_i + C) = 1$. Thus $\deg L^1 j^* \omega_C = 0$ and $\deg \mathcal{F}_C|_{E_i} = -2$. The sheaf $\mathcal{F}_C|_{E_i}$ is free on E_i of rank two, degree -2 and no global sections, thus

$$\mathcal{F}_C|_{E_i} = \mathcal{O}_{E_i}(-1)^{\oplus 2}.$$

□

Now from (1) we tensor the first column by $\mathcal{O}_{\tilde{S}}(-E)$

$$0 \rightarrow \varepsilon^* T_S(-C - E) \rightarrow \mathcal{F}_C(-E) \xrightarrow{\tau} T_C \left(-\sum_{i=1}^{\delta} p_i + q_i \right) \rightarrow 0$$

Proposition 1.9. *The map*

$$H^1(\tau) : H^1(\tilde{S}, \mathcal{F}_C(-E)) \rightarrow H^1(C, T_C(-p_1 - q_1 - \dots - p_{\delta} - q_{\delta}))$$

is the differential of $c : \mathcal{V}_{g,\delta} \rightarrow \mathcal{M}_{g-\delta, [2\delta]}$.

Proof. The blow up map $\varepsilon : \tilde{S} \rightarrow S$ restricted to C is finite so the higher derived images of ε_* are always trivial and $\varepsilon_* T_C(-\sum p_i + q_i)$ is the tangent sheaf of the nodal curve X . On the other hand if we apply ε_* to the second row in diagram (1) we get that $\varepsilon_* \mathcal{F}_C \cong T_S \langle X \rangle$ (see proof in [FKPS08, Prop. 4.22]) and

$$R^i \varepsilon_* \mathcal{F}_C \cong R^i \varepsilon_* \varepsilon^* T_S = 0 \text{ for } i > 0.$$

Then from the exact sequence

$$0 \rightarrow \mathcal{F}_C(-E) \rightarrow \mathcal{F}_C \rightarrow \mathcal{O}_E(-1)^{\oplus 2} \rightarrow 0$$

we can conclude that $R^i \varepsilon_* \mathcal{F}_C(-E) = 0$ for $i > 0$. There are natural isomorphisms coming from the Leray Spectral Sequence fitting in the diagram

$$\begin{array}{ccc} H^1(\tilde{S}, \mathcal{F}_C(-E)) & \xrightarrow{H^1(\tau)} & H^1(C, T_C(-p_1 - p_2)) \\ \downarrow \cong & & \downarrow \cong \\ H^1(S, \varepsilon_* \mathcal{F}_C(-E)) & \xrightarrow{H^1(\varepsilon_* \tau)} & H^1(T_X). \end{array}$$

We claim that the map at the bottom factors through a natural isomorphism $H^1(\varepsilon_* \mathcal{F}_C(-E)) \cong H^1(T_S \langle X \rangle)$ and H^1 of the restriction map $T_S \langle X \rangle \rightarrow T_X$ sending locally trivial first order deformations of the pair (S, X) to nodal deformations of X .

We proved in Proposition 1.8 that the restriction of \mathcal{F}_C to E is isomorphic to $\mathcal{O}_E(-1)^{\oplus 2}$ and $\varepsilon_* \mathcal{F}_C \cong T_S \langle X \rangle$. On the other hand as a general fact, if $N = \text{Sing}(X)$, the kernel of the surjection $\varepsilon^* \mathcal{I}_N \rightarrow \mathcal{I}_E$ is $L^1 \varepsilon^* \mathcal{O}_E \cong \mathcal{O}_E(-1)$. Moreover

$$\varepsilon_*(\mathcal{F}_C \otimes \mathcal{O}_E(-1)) = 0 \text{ and } R^1 \varepsilon_*(\mathcal{F}_C \otimes \mathcal{O}_E(-1)) \cong H^1(E, \mathcal{O}_E(-2)^{\oplus 2}) \otimes \mathcal{O}_N.$$

Using this one can see that the sequence

$$(2) \quad 0 \rightarrow \varepsilon_*(T_S \langle X \rangle \otimes \varepsilon^* \mathcal{I}_N) \rightarrow \varepsilon_* \mathcal{F}_C(-E) \rightarrow H^1(E, \mathcal{O}_E(-2)^{\oplus 2}) \otimes \mathcal{O}_N \rightarrow 0$$

is exact. Since $\mathcal{F}_C \otimes \mathcal{O}_E(E) \cong \mathcal{O}_E(-2)^{\oplus 2}$ and $H^0(T_S \langle X \rangle) = H^2(T_S \langle X \rangle) = 0$, from 2 we have

$$\begin{array}{ccccc} & & H^1(E, \mathcal{O}_E(-2)^{\oplus 2}) & & \\ & \nwarrow \cong \text{dashed} & \downarrow & & \\ H^0(T_S \langle X \rangle \otimes \mathcal{O}_N) & \hookrightarrow & H^1(\varepsilon_*(\mathcal{F}_C \otimes \varepsilon^* \mathcal{I}_N)) & \xrightarrow{\beta} & H^1(T_S \langle X \rangle) \\ & & \downarrow \alpha & & \downarrow \\ & & H^1(\varepsilon_* \mathcal{F}_C(-E)) & \xrightarrow{H^1(\varepsilon_* \tau)} & H^1(T_X). \end{array}$$

Notice that

$$H^0(S, T_S \langle X \rangle \otimes \mathcal{O}_N) \cong H^1(E, \mathcal{O}_E(-2)^{\oplus 2}) \cong \mathbb{C}^{2\delta}.$$

Moreover the map β is the one coming from $\varepsilon^* \mathcal{I}_N \rightarrow \mathcal{O}_S$ after tensoring and pushing down to S and the map α is the one coming from the same the surjection $\varepsilon^* \mathcal{I}_N \rightarrow \mathcal{I}_E$. This give us a

natural isomorphism fitting in the diagram

$$\begin{array}{ccc} H^1(\varepsilon_* \mathcal{F}_C(-E)) & \xrightarrow{\cong} & H^1(T_S \langle X \rangle) \\ & \searrow H^1(\varepsilon_* \tau) & \downarrow \\ & & H^1(T_X) \end{array}$$

□

Recall that as stated in [FKPS08] if $V_{g,\delta} \rightarrow \mathcal{V}_{g,\delta}$ is an étale atlas and $\mathcal{X} \hookrightarrow \mathcal{S}$ is the universal family of pairs (X, S) induced by it, there is a universal embedded resolution

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \tilde{\mathcal{S}} \\ \downarrow \rho & & \downarrow \varepsilon \\ \mathcal{X} & \hookrightarrow & \mathcal{S} \\ \downarrow & \nearrow & \\ V_{g,\delta} & & \end{array}$$

where ε is the blow up of \mathcal{S} along the nodal locus $\mathcal{N}_\delta \subset \mathcal{X} \hookrightarrow \mathcal{S}$ and ρ is fiber-wise over $V_{g,\delta}$ the normalization. Every locally trivial first order deformations of a pair (X, S) induce one of the pair (C, \tilde{S}) parametrized by $H^1(\tilde{S}, \mathcal{F}_C)$.

Proof of Theorem 0.5. We will prove that the differential is onto. Indeed,

$$dc : H^1(\tilde{S}, \mathcal{F}_C(-E)) \rightarrow H^1(C, T_C(-p_1 - q_1 - \dots - p_\delta - q_\delta))$$

is surjective if

$$H^2(\tilde{S}, \varepsilon^* T_S(-C - E)) = 0.$$

We call H the polarization $\mathcal{O}_S(X)$. By Serre duality and since $\varepsilon^* H = C + 2E$,

$$H^2(\tilde{S}, \varepsilon^* T_S(-C - E)) \cong H^0(\tilde{S}, \varepsilon^* \Omega_S^1(H)).$$

But

$$R\varepsilon_*(\varepsilon^* \Omega_S^1(H)) \cong \Omega_S^1(H) \otimes R\varepsilon_* \mathcal{O}_{\tilde{S}}$$

and since ε is birational $R\varepsilon_* \mathcal{O}_{\tilde{S}} \cong \mathcal{O}_S$, i.e., the higher direct images $R^i \varepsilon_* \mathcal{O}_{\tilde{S}}$ vanish for $i > 0$,

$$H^0(\tilde{S}, \varepsilon^* \Omega_S^1(H)) \cong H^0(S, \Omega_S^1(H))$$

and this last group is trivial when $g \leq 9$ or $g = 11$. When $g = 10$, $H^0(S, \Omega_S^1(H)) \cong \mathbb{C}$, i.e., the image is divisorial. See [Bea04, §5.2]. To compute the class of this divisor in $\text{Pic}_{\mathbb{Q}}(\mathcal{M}_{10-\delta, [2\delta]})$ we just need to pull back the class $\overline{\mathcal{K}}$ of the closure of the K3 locus in \mathcal{M}_{10} by the boundary map $\xi : \overline{\mathcal{M}}_{10-\delta, 2\delta} \rightarrow \overline{\mathcal{M}}_{10}$ and then push it down by the $S_{2\delta}$ -quotient $\pi : \overline{\mathcal{M}}_{10-\delta, 2\delta} \rightarrow \overline{\mathcal{M}}_{10-\delta, [2\delta]}$,

$$c_*[\mathcal{V}_{10,\delta}] = \frac{1}{n!} \pi_* (\xi^* \overline{\mathcal{K}}).$$

If we restrict it to $\mathcal{M}_{10-\delta, [2\delta]}$ we have our result. □

Remark 1.10. Take for example $g = 10$ and $\delta = 1$. Even though the map to the normalization \mathcal{M}_9 is surjective, the map to $\mathcal{M}_{9,[2]}$ is divisorial, i.e., if C is a general genus 9 curve there is a codimension one cycle in the symmetric product without the diagonal $\Gamma \subset C^{[2]} \setminus \Delta$ consisting of points $p + q$ such that after identifying them, the nodal curve $C/p \sim q$ lies on a $K3$. Let $C^{[2]} \setminus \Delta$ be a general fiber of $\pi : \overline{\mathcal{M}}_{9,[2]} \rightarrow \overline{\mathcal{M}}_9$, since the complex structure of the curve along the fiber is constant, the Hodge bundle restricts to the trivial bundle and the class of Γ in $\text{Pic}(C^{[2]} \setminus \Delta)$ is given by

$$\frac{7\lambda + \psi_1 + \psi_2}{2} \cdot \pi^*(\text{pt}) = K_C + C.$$

The same argument works for $\delta \geq 1$.

2. QUADRATIC DIFFERENTIALS

Let S be the blow-up of \mathbb{P}^2 along $0 \leq r \leq 8$ many point in general position. The surface S is a del Pezzo surface and the class $-2K_S$ is ample. There is a moduli of such surfaces \mathcal{P}_r realized as the quotient of an open \mathcal{U} of $(\mathbb{P}^2)^r$ by the group $PGL(3)$. The moduli space has dimension $\min\{2r - 8, 0\}$ and over it sits a natural space

$$\mathcal{B}_r = \{(S, C) \mid S \in \mathcal{P}_r \text{ and } C \in |-2K_S| \text{ smooth and irreducible}\}$$

with fibers over each del Pezzo surface $S \in \mathcal{P}_r$ open subsets of the projective space $|-2K_S|$. Since $\chi(\mathcal{O}_S) = 1$ and by Riemann-Roch and Kodaira Vanishing

$$\dim H^0(S, \mathcal{O}_S(-2K_S)) = \chi(\mathcal{O}_S) + 3K_S^2 = 28 - 3r.$$

The fiber dimension of the map $\mathcal{B}_r \rightarrow \mathcal{P}_r$ is $27 - 3r$ and the dimension of \mathcal{B}_r is $19 - r$. On the other hand if $C \in |-2K_S|$ is a smooth irreducible curve on S , the genus of C satisfies

$$2g - 2 = C^2 + K_S \cdot C = 2K_S = 18 - 2r$$

and there is a natural map

$$\begin{aligned} \psi_r : \mathcal{B}_r &\rightarrow \mathcal{M}_{10-r} \\ (S, C) &\mapsto [C]. \end{aligned}$$

Proposition 2.1. *When $4 \leq r \leq 7$ the map ψ_r is dominant.*

Proof. Let $[C] \in \mathcal{M}_g$ be a general smooth curve with $3 \leq g \leq 6$. The Brill-Noether number $\rho(g, 2, 6) \geq 0$ and the general curve $[C] \in \mathcal{M}_g$ has a plane nodal model $\Gamma \subset \mathbb{P}^2$ of degree 6 with $10 - g$ nodes. Take $r = 10 - g$, $S = \text{Bl}_r \mathbb{P}^2$ the blow-up of \mathbb{P}^2 along the nodes, E_1, \dots, E_r the exceptional divisors and L the proper transform of the line. Then the proper transform of Γ is smooth and lies in the linear system $-2K_S = 6L - 2E_1 - \dots - 2E_r$. \square

Lemma 2.2. *Let $\nu = (n_1, \dots, n_{m+g})$ be a partition of $4g - 4$ of length $m + g$ with at least one non-even entry. The forgetful map*

$$\begin{aligned} \mathcal{Q}(\nu) &\rightarrow \mathcal{M}_{g,m} \\ [C, p_1, \dots, p_{m+g}] &\mapsto [C, p_1, \dots, p_m]. \end{aligned}$$

is dominant.

Proof. The diagram

$$\begin{array}{ccc} \mathcal{Q}(\nu) & \xrightarrow{i} & \mathcal{M}_{g,n+g} \\ \downarrow \pi & & \downarrow \sigma_\nu \\ \mathcal{M}_{g,n} & \xrightarrow{c} & \mathcal{J}_n^{2 \cdot (2g-2)} \end{array}$$

is cartesian, where \mathcal{J}_n^{4g-4} is the universal jacobian of degree $4g-4$ over $\mathcal{M}_{g,n}$, the map c is the 2-canonical section and σ_ν is the global Abel-Jacobi map given by

$$\sigma_\nu : [C, p_1, \dots, p_{m+g}] \mapsto \left[C, p_1, \dots, p_m, \mathcal{O}_C \left(\sum_{i=1}^{m+g} n_i p_i \right) \right].$$

For a smooth curve with marked points $[C, p] \in \mathcal{M}_{g,m}$ we fix

$$L_{[C,p]} = \mathcal{O}_C \left(\sum_{i=1}^m n_i p_i \right).$$

For dimension reasons the locus of curves $[C, p] \in \mathcal{M}_{g,m}$ such that $\omega^{\otimes 2} - L_{[C,p]}$ is supported at less than g points is of codimension at least one. Then if $[C, p]$ is general in $\mathcal{M}_{g,m}$, the image of the restriction σ_ν to the fiber of $\mathcal{J}_n^{4g-4} \rightarrow \mathcal{M}_{g,m}$ over $[C, p]$,

$$\begin{aligned} \sigma_\nu : C^{\times g} \setminus \Delta &\rightarrow \text{Pic}^{4g-4}(C) \\ (p_{m+1}, \dots, p_{m+g}) &\mapsto L_{[C,p]} + \mathcal{O}_C(n_{m+1}p_{m+1} + \dots + n_{m+g}p_{m+g}), \end{aligned}$$

contains $\omega^{\otimes 2}$. Therefore, the Abel-Jacobi map σ_ν dominates the image of the 2-canonical section c . \square

Proof of Theorem 0.3. Let $g \leq 6$ and ν a partition of $4g-4$ with at least one non-even entry and length $l(\nu) \geq g$. Let $[C] \in \mathcal{Q}(\nu)$ be a general curve on the strata, then C is general in moduli so we can assume lies on the image of ψ_r with $r = 10 - g$. As in Section 1.1, the linear system $|-2K_S|$ embeds S in \mathbb{P}^{3g-3} and realizes C as an hyperplane section

$$C = S \cap H \subset \mathbb{P}^{3g-3}.$$

The restriction of $-2K_S$ to C is $2K_C$ thus the map $S \hookrightarrow \mathbb{P}^{3g-3}$ restricted to C is the 2-canonical embedding and since $\sum n_i x_i \in \text{Div}(C)$ is a quadratic differential there must be a codimension 2 hyperplane $\Lambda_\mu \subset \mathbb{P}^{3g-3}$ with

$$\Lambda_\mu \cdot S = \sum n_i x_i.$$

Again let P be the pencil of hyperplanes $H \in (\mathbb{P}^{3g-3})^\vee$ containing Λ_μ . The points x_1, \dots, x_n lie on the base locus of this pencil and there is a rational map

$$\begin{aligned} P &\dashrightarrow \mathcal{Q}(\nu) \\ H' &\mapsto [S \cap H', x_1, \dots, x_n]. \end{aligned}$$

Is left to prove that the map is non-trivial. Let $\pi : \mathcal{U} \rightarrow \mathbb{P}^1$ be the family of curves induced by the pencil P , Z the base locus of P and $\varepsilon : \mathcal{U} \rightarrow S \rightarrow \mathbb{P}^2$ the composition of the blow-up of S at Z . with the blow up map $\varepsilon : S \rightarrow \mathbb{P}^2$. If every fiber of π is smooth then the topological Euler characteristic of \mathcal{U} is

$$\chi(\mathcal{U}, \mathbb{Z}) = \chi(\mathbb{P}^1, \mathbb{Z}) \cdot \chi(\pi^{-1}(\text{point}), \mathbb{Z}) = 2 \cdot (2 - 2g).$$

But \mathcal{U} is the composition of the blow up of \mathbb{P}^2 at r points together with the blow up at $|Z|$ points on S . Thus,

$$\chi(\mathcal{U}, \mathbb{Z}) = 3 + r + |Z| \geq 3$$

which is a contradiction. Thus $\pi : \mathcal{U} \rightarrow \mathbb{P}^1$ must have singular fibers. This proves non-isotriviality. Fibers of π might still be curves isomorphic to C with a rational tail attached in which case the moduli map induced by the pencil is still trivial. But this cannot happen since if we see the pencil as a \mathbb{P}^1 -family of r -nodal sextics in \mathbb{P}^2 , if $C \sim R + C'$ where R is an irreducible rational tail, then R a line or a conic on \mathbb{P}^2 in which case the residual curve C' drops in genus. \square

3. UNIRATIONALITY IN SMALL GENUS

Recall that a variety is unirational if its dominated by a rational variety. As above $\mu = (m_1, \dots, m_n)$ is an holomorphic partition of $2g - 2$. When the length of the partition is $g - 1$ and $\mathcal{H}_g(\mu)$ is connected (i.e. $\mu \neq (2, \dots, 2)$ and $g \geq 3$ see [KZ03]) then a similar argument as in [FV14] can be used to prove unirationality for $g \leq 6$.

Proof. We will construct a projective bundle \mathcal{P}_g over a rational variety \mathcal{U} that dominates $\overline{\mathcal{H}}_g(\mu)$. Let $\mathcal{U} \subset (\mathbb{P}^2)^{N_g}$ be the open set parametrizing general N_d -tuples of points with

$$N_d = \binom{d-1}{2} - 1 \quad \text{and } d \text{ such that } \rho(g, 2, d) \geq 0.$$

We fix

$$\delta = \binom{d-1}{2} - g$$

and $(\bar{x}, \bar{y}) = (x_1, \dots, x_\delta, y_1, \dots, y_{g-1}) \in \mathcal{U}$. The fiber of \mathcal{P}_g over (\bar{x}, \bar{y}) is going to be the linear space $V_{(\bar{x}, \bar{y})}$ of plane nodal curves $\Gamma \in |\mathcal{O}_{\mathbb{P}^2}(d)|$ of degree d singular along \bar{x} and having μ -contact at \bar{y} with the unique degree $d - 3$ curve $E_{(\bar{x}, \bar{y})}$ passing through the $\binom{d-1}{2} - 1$ points $x_1, \dots, x_\delta, y_1, \dots, y_{g-1}$, i.e.,

$$\Gamma \cdot E_{(\bar{x}, \bar{y})} = \sum m_i y_i + D$$

where $\text{Supp}(D) = \text{Sing}(\Gamma)$. To be more explicit consider the *Severi variety*

$$U_{d,g} := \left\{ [\Gamma \hookrightarrow \mathbb{P}^2] \mid \deg(\Gamma) = d, \Gamma \text{ is nodal irreducible and } p_a(\Gamma) = g \right\}$$

and our incidence variety

$$\mathcal{P}_g := \left\{ \left([\Gamma \hookrightarrow \mathbb{P}^2], (\bar{x}, \bar{y}) \right) \in U_{d,g} \times \mathcal{U} \mid \bar{x} = \text{Sing}(\Gamma), \Gamma \cdot E_{(\bar{x}, \bar{y})} = \sum m_i y_i + D \right\}.$$

There are two natural maps

$$\begin{array}{ccc} & \mathcal{P}_g & \\ \nu_g \swarrow & & \searrow \pi_2 \\ \mathcal{M}_{g,g-1} & & (\mathbb{P}^2)^{N_d}, \end{array}$$

Where ν_g maps $(\Gamma, \bar{x}, \bar{y})$ to $[C, \bar{y}]$ with C the normalization of the nodal curve $\nu : C \rightarrow \Gamma$ and we identify \bar{y} with the pre-image under normalization. Notice that $E_{(\bar{x}, \bar{y})} \in |\mathcal{O}_{\mathbb{P}^2}(d-3)|$ and if we restrict and pull back by normalization

$$\mathcal{O}_C \left(\sum m_i y_i \right) \cong \nu^* \mathcal{O}_\Gamma(d-3) \left(-\nu^{-1}(\bar{x}) \right)$$

and by adjunction

$$\sum m_i y_i \sim K_C.$$

Thus the image of ν_g lies in $\mathcal{H}_g(\mu)$. Moreover it is dominant, indeed if $[C, \bar{y}]$ is a general point in $\mathcal{H}_g(\mu)$, then since $\rho(g, 2, d) \geq 0$, and choosing $[C, \bar{y}]$ and $A \in G_d^2(C)$ general we might assume that the associated map $\phi_A : C \rightarrow \mathbb{P}^2$ realize C as δ -nodal curve and again by adjunction, since $\sum m_i y_i$ is canonical

$$\nu^* \mathcal{O}_\Gamma(d-3) \left(-\nu^{-1}(\bar{x}) - \sum m_i y_i \right)$$

has a unique section which coincides with $\Gamma \cdot E_{(\bar{x}, \bar{y})}$, where $E_{(\bar{x}, \bar{y})}$ is the unique degree $d-3$ curve passing through the N_d -tuple in linear general position $\{\bar{x}, \bar{y}\}$. Notice that

$$\dim \mathcal{P}_g = 3g - 3 + \rho(g, 2, d) + \dim(\mathrm{PGL}(3)) = 3d + g - 1$$

and a necessary condition for π_2 to be dominant is

$$\dim \mathcal{P}_g \geq 2N_d.$$

This together with $\rho(g, 2, d) \geq 0$ forces g to be smaller or equal than 6. The non-emptiness of the general fiber of π_2 follows from the fact that the amount of linear conditions on the projective space $|\mathcal{O}_{\mathbb{P}^2}(d-3)|$ is less than the dimension.

Finally notice that entire argument holds for $m_i \geq 0$. Imposing some entries to be zero give us our result for all partitions $l(\mu) \leq g-1$. \square

4. NON-IRREDUCIBLE STRATA

Let $\mu = (2, \dots, 2)$ be a partition of $2g-2$. There is a natural map from strata to the space of spin curves which we already know that splits into two connected components depending on the parity of the theta characteristic,

$$\begin{aligned} \pi : \mathcal{H}_g(\mu) &\rightarrow \mathcal{S}_g^+ \amalg \mathcal{S}_g^- \\ [C, x_1, \dots, x_{g-1}] &\mapsto [C, \mathcal{O}_C(x_1 + \dots + x_{g-1})]. \end{aligned}$$

A general point $[C, \nu] \in \mathcal{S}_g^+$ satisfies $h^0(\nu) = 0$ and the condition $h^0(\nu) \geq 2$ is divisorial. The divisor $x_1 + \dots + x_{g-1}$ is effective and for a general point on the image of $\mathcal{H}^+ \rightarrow \mathcal{S}_g^+$ we have that $h^0(x_1 + \dots + x_{g-1}) = 2$. See [Far10]. Thus $\overline{\mathcal{H}}^+/S_{g-1}$ is a \mathbb{P}^1 -bundle over some open of a divisor Z in \mathcal{S}_g^+ . In particular for every genus, through a general point of $\overline{\mathcal{H}}^+/S_{g-1}$ passes a rational curve.

4.1. Hyperelliptic Strata. Let $\mathcal{H}_{g,1} \subset \mathcal{M}_{g,1}$ be the space of pairs $[C, p]$ where C is an hyperelliptic curve and $p \in C$ is a weierstrass point on it. Can be defined as the fiberproduct over \mathcal{M}_g of the weierstrass divisor \mathcal{W} and the hyperelliptic locus $\mathrm{Hyp}_{g,1} \subset \mathcal{M}_{g,1}$

$$\mathcal{H}_{g,1} = \mathrm{Hyp}_{g,1} \times_{\mathcal{M}_g} \mathcal{W}.$$

Recall that the space $\mathrm{Hyp}_g \subset \mathcal{M}_g$ is birational to $\mathcal{M}_{0,2g+2}/S_{2g+2}$. In particular there is a dominant map

$$\mathcal{M}_{0,2g+2} \rightarrow \mathcal{H}_{g,1}$$

sending the tuple of points (p_1, \dots, p_{2g+2}) on \mathbb{P}^1 to the unique double cover ramified over $p_1 + \dots + p_{2g+2}$

$$f : C \rightarrow \mathbb{P}^1$$

together with the weierstrass point $f^*(p_1)$. This give us unirationality of $\mathcal{H}_{g,1}$. On the other hand is easy to see that p is a Weierstrass point on an hyperelliptic curve C if and only if $(2g-2)p \sim K_C$, i.e.,

$$\mathcal{H}_{g,1} = \mathcal{H}_g^{\text{hyp}}(2g-2).$$

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